ON THE FREQUENCY DISTRIBUTION DENSITY OF OSCILLATIONS OF A THIN SHELL

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We study the problems concerning the frequency distribution density of free linear oscillations of a thin shell. Particular attention is given to the effect of concentration of frequencies at certain critical points, which was noted in [1−5]. We also estimate the domain of applicability of the asymptotic method of determining the free oscillations of a shell [5−11].

In [1−5] the frequency distribution density is studied with the aid of the formula

$$\omega^2 = \frac{D}{\rho h}\left[(k_1^2 + k_2^2)^2 + \frac{Eh}{DR_1^2}\frac{(k_1^2\chi + k_2^2)^2}{(k_1^2 + k_2^2)^2}\right] = \frac{D}{\rho h}\left[r^4 + \frac{Eh}{DR_1^2}(\chi\cos^2\varphi + \sin^2\varphi)^2\right] \quad (1)$$

$$\chi = \frac{R_1}{R_2}, \qquad \left(r^2 = k_1^2 + k_2^2,\ \varphi = \operatorname{arctg}\frac{k_2}{k_1}\right) \quad (2)$$

where ω is the frequency of free oscillation of the shell, D is cylindrical rigidity, ρ is the material density, h is thickness, E is the Young's modulus, k_1 and k_2 are wave numbers, and R_1 and R_2 denote the principal radii of curvature of the surface.

In (1) the quantities R_1 and R_2 must of course be taken as constant. We shall therefore assume that the quantity χ in (2) satisfies the inequalities

$$-1 \leqslant \chi \leqslant 1 \quad (3)$$

(for which the constants must be suitably numbered).

Following [1−5] let us set

$$\omega_1 = \left(\frac{E}{\rho}\right)^{1/2}\frac{1}{R_2}, \qquad \omega_2 = \left(\frac{E}{\rho}\right)^{1/2}\frac{1}{R_1} \qquad (\omega_1 \leqslant \omega_2) \quad (4)$$

and write (1) in the form

$$\omega^2 = \omega'^2 + \omega''^2,\ \omega' = \left(\frac{D}{\rho h}\right)^{1/2}(k_1^2 + k_2^2), \qquad \omega'' = \omega_2\frac{k_1^2\chi + k_2^2}{k_1^2 + k_2^2} \quad (5)$$

The physical meaning of the notation introduced is obvious; ω' is the oscillation frequency of the equivalent plate, i.e. of the plate whose dimensions, cylindrical rigidity and mass per unit area are identical with those of the shell, and ω'' is the oscillation frequency of the shell as a membrane. The effect of the curvature of the shell is reflected only in ω'', and the relative contribution of this quantity decreases with increasing ω. We can therefore divide the frequency spectrum defined by (1) into two parts: one (small ω) in which the influence of curvature is relatively large and the other (large ω) in which this influence is insignificant.

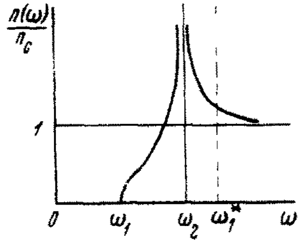

Fig. 1

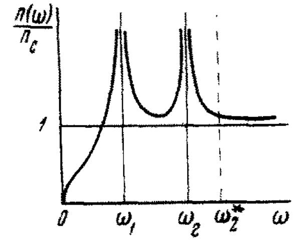

Fig. 2

It can easily be shown that for a shell of positive curvature we have

$$\omega_1 \leqslant \omega'' \leqslant \omega_2 \quad (6)$$

while for that of zero and negative curvature we have

$$0 \leqslant \omega'' \leqslant \omega_2$$

We can therefore assume that the division of the spectrum occurs at $\omega = \omega_2^*$, where ω_2^* is slightly larger than ω_2. This assumption is confirmed by the graphs shown in Figs. 1 and 2 depicting the oscillation frequency density of the shells, obtained in [1– 5]. The horizontal lines appearing on the graphs correspond to the eqilvalent plate.

Formula (1) was obtained by so called asymptotic method [5–11] based on a number of assumptions. For the time being we shall note two of them:

1) edges of the shell follow the lines of curvature x_1 = const and x_2 = const forming a curvilinear rectangle and

2) the dynamic edge effect is not degenerate.

The name of dynamic (nondegenerate) edge effect was assigned in [5–11] to the stress-strain state appearing at the shell edge during the oscillations, and decaying rapidly with increasing distance from the edge. When the effect, instead of decaying, becomes oscillatory, we call it the degenerate dynamic edge effect.

We shall now obtain the conditions of degeneracy.

Let us keep the value of ω in (1) fixed and consider this expression as an equation of some curve γ_ω (level line) on the Cartesian k_1k_2 plane. Fig. 3 shows the shape assumed by the curves γ_ω in the first quadrant when $\chi > 0$ and $\chi < 0$, and ω satisfies the following inequalities:

$$(1)\ \omega \leqslant \omega_1,\ (2)\ \omega_1 < \omega < \omega_2$$
$$(3)\ \omega_2 < \omega < \Omega,\ (4)\ \Omega < \omega \quad (7)$$

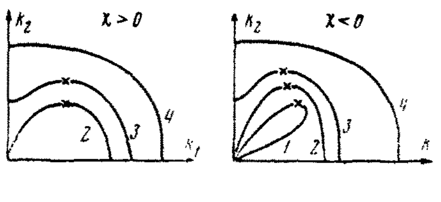

Fig. 3

(here the numbers in brackets denote the curves shown on Fig. 3. Curve 1 becomes imaginary when $\chi > 0$).

In the above inequalities ω_1 and ω_2 are given by the formulas (4), while the meaning of Ω is explained below.

It can easily be shown that, when ω is given and γ_ω intersects any straight line k_2 = = const twice in the first quadrant, the dynamic edge effect degenerates at the line x_1 = const (*). Having noted that, we shall take Ω in (7) to be the least value of ω at which γ_ω intersects any straight line k_2 = const in the first quadrant once only. Then the condition of degeneracy can be written as $\omega < \Omega$. The dynamic edge effect is now absent or present, depending whether the latter inequality holds or not.

Let us denote by λ the set of values of ω for which k_2 (as the function of k_1) has a local maximum in the first quadrant (the points marked with × in Fig. 3). Then Ω will be the upper limit of λ.

To determine Ω, let us consider the formula

$$\frac{dk_2}{dk_1} = -\frac{\partial\omega^2 / \partial k_1}{\partial\omega^2 / \partial k_2}$$

*) This must have escaped the notice of Bolotin, since he includes γ_ω corresponding to the degeneration of the edge effect on figures appearing in all his papers dealing with the frequency distribution density.

Since the denominator in its right-hand side is bounded, the maximum value of k_2 is reached at the points at which

$$\frac{\partial \omega^2}{\partial k_1} = \frac{1}{r^2}\frac{Dk_1}{\rho h}\left[r^4 - \frac{Eh}{DR_1^2}(1-\chi)(\chi\cos^2\varphi + \sin^2\varphi)\sin^2\varphi\right] = 0$$

Using this to express r in therms of φ and inserting the resulting expression into (1), we obtain

$$\lambda = \omega_2 (\chi\cos^2\varphi + \sin^2\varphi)^{1/2} (2\sin^2\varphi + \chi\cos 2\varphi)^{1/2}$$

from which we easily see that the upper limit of λ is reached at $\varphi = \pi / 2$ and is equal to $\omega_2\sqrt{2-\chi}$. We can therefore write the condition of degeneracy of the dynamic edge effect in form of the following inequality:

$$\omega < \omega_2\sqrt{2-\chi} \tag{8}$$

This means that the initial part of the spectrum where (8) holds, contains frequencies at which the dynamic edge effect does not exist, i. e. the asymptotic method becomes (as noted in [5–11]) unsuitable. Violation of the inequality (8) implies the existence of the dynamic edge effect and validity (from this point of view) of the asymptotic method for computing the corresponding frequencies. The quantity appearing in the right-hand side of (8) exceeds slightly ω_2. It can be tentatively identified with ω_2* and this means, that the dynamic edge effect exists only in this part of the spectrum, which is weakly dependent on the curvature of the shell.

If $\chi = 1$ (a sphere), the dynamic edge effect does not degenerate at $\omega < \omega_2$. Moreover, from (4)–(6) it follows that this inequality holds for all frequencies in the case of a spherical shell (*). Consequently, in the latter case the dynamic edge effect always exists. This also follows from the fact that the curves γ_ω degenerate at $\chi = 1$ in the circle of $r = \text{const}$. Since γ_ω assume this shape also when $R_1 = R_2 = \infty$, it follows that no degeneration occurs in a plate. We can also show that the dynamic edge effect will degenerate at one of the two pairs of edges, within the initial part of the spectrum, for any other shell.

In [6] the process of investigating the degeneracy of the edge effect was made unnecessarily complicated. Bolotin showed that the degeneracy is impossible for the spherical shell and the plate, but did not observe that no other shells with this property exists.

When deriving (1), additional assumptions were made, namely, that a constant metric can be established on the neutral surface and, that R_1 and R_2 are constant.

As we know, the constant metric is realizable only on a surface of zero curvature, and constant curvatures R_1 and R_2 can only be found on a sphere, a circular cylinder and a plane. Consequently, formulas (1) can only be used for plates, shallow shells and circular cylindrical shells (if the wave numbers are large enough, then the rise of the cylindrical shell can have any value).

The behavior of a sufficiently large number of frequencies is characterized by their distribution density. There are relatively few frequencies corresponding to oscillations accompanied by small variations in the stress-strain state. Hence we can justifiably

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*) Obviously we are discussing the frequencies which can be investigated using the asymptotic method, i. e. the frequencies in a spherical shell, rectangular in one plane. If, e. g., the shell has the form of a spherical segment, then some of its frequencies will fall to the left of ω_2 and their number will increase as the shell thickness decreases.

disregard the errors occurring in the frequencies corresponding to small variations, when computing the frequency distribution density. Formula (1) can now be regarded as one yielding sufficient accuracy only in calculating the oscillation frequencies corresponding to sufficiently large variations.

Large variations in the stress-strain state imply the appearance of a dense grid of nodal lines. Although these lines partition the shell into flat segments, this does not mean that formula (1) becomes suitable for a shell of any form and rise.

In a shell as a whole, R_1 and R_2 may vary over a wide range and we must know, taking into account the conditions of the problem, how to compute these constant mean values of R_1 and R_2 which must be inserted into (1). Until such a method is available, the relation (1) will give rise to error independent of the number of frequency. At the same time the following relations hold

$$\lim \frac{\omega_{(n)}}{\omega_{(n+1)}} = 1, \quad \lim \omega_n = \infty \quad (n \to \infty)$$

from which it follows that at sufficiently large number of frequency n, any relative error independent of n will give rise to an absolute deviation from $\omega_{(n)}$ by an amount exceeding $\omega_{(n+1)} - \omega_{(n)}$. In other words, formula (1) becomes more and more erroneous with increasing n, and at sufficiently large n it ceases to resolve the neighboring frequencies.

It follows that the conditions of applicability of the asymptotic method are:

1) sufficiently small characteristic wavelengths of the oscillation mode;

2) almost constant metric of the neutral surface;

3) absence of the wave numbers k_1 and k_2 from the regions of degeneracy of the dynamic edge effect;

4) almost constant curvatures of the neutral surface.

The first three conditions were given in [5] ch. 8, while the fourth condition follows from the foregoing considerations. Only shallow shells and sufficiently short circular cylindrical shells satisfy all the above conditions, and even then an exception must be made for oscillations arising from the curvature.

Let us now turn our attention to the results obtained from the frequency distribution densities. These are depicted on graphs (Figs 1 and 2), referring to the shells of positive and negative curvature respectively. In both cases the ratio of the frequency distribution densities of the shell and of the equivalent plate computed by the Courant's [12] method, is plotted on the ordinate.

Naturally, the only interesting parts of the graphs are those for which $\omega < \omega_2^*$, since the inequality (8) is violated in these regions, i.e. the dynamic edge effect degenerates and formula (1) becomes invalid for an arbitrarily supported shell.

Thus we have failed to show that the graphs (Fig. 1 and 2) can be referred to arbitrarily supported shells. The graphs were based on (1) which was derived in [5–11] by means of the asymptotic method, and the latter breaks down in the most interesting parts of the graph (except for a sphere). We must therefore treat Eq. (1) as a formula, known for a long time, for computing the frequencies of oscillations of a hinged shell. The asymptotic method is not required for its derivation and Figs. 1 and 2 are, strictly speaking, valid only for this case.

At $\omega = \omega_2$ the graphs have an infinite discontinuity, indicating a substantial concentration of frequencies. Let us discuss this phenomenon.

Let a shallow shell approach the limiting plate infinitely near, without however attaining the limit. Then the graphs of the frequency distribution density of the shells will approach the graph depicting the frequencies of the plate. This however will not affect the infinite discontinuity at $\omega = \omega_2$ and this produces a result which is physically inconsistent (concentration of frequencies in a plate of arbitrarily low curvature). The inconsistency is caused by the fact that the more flattened the shell, the less frequencies appear in the interval $\omega < \omega_2^*$, the concept of density therefore, as defined by Courant, losing its meaning.

This constitutes another limitation restricting the range of applicability of the graphs in question, namely that a shell of fixed thickness should not be too shallow.

Let us estimate the number of frequencies falling within the interval $\omega < \omega_2^*$. We have shown before that this is roughly equal to the number of frequencies corresponding to the degenerate dynamic edge effect. The latter was computed in [6], where the following inequality was obtained for a square cylindrical panel:

$$m^2 + n^2 < \frac{a^2}{\pi^2 hR} \sqrt{12(1-\mu^2)} \tag{9}$$

where a denotes the side of the square, R is the radius of the cylinder, μ is the Poisson's ratio, m and n are positive integers, each pair corresponding to some oscillation frequency of the shell.

When $a / h = 100$, $R / a = 1$ and $\mu = 0.25$ (a thin, not very shallow shell), the right-hand side of (9) is equal to 34.1; this in turn yields the required number of frequencies, namely 22. This is hardly sufficient for constructing a graph of complexity comparable to that of Fig. 2.

From this point of view, the graphs in question are of practical use only in the case of very thin shells, or shells with sufficient rise. But in the latter case R_1 and R_2 in (1) will vary over wide limits, the concept of critical frequencies ω_1 and ω_2 will itself become indefinite and the infinite discontinuities will be smeared out. The latter process can be described qualitatively, by leaning closer towards Courant's ideas [12].

Let us subdivide the shell into elementary segments, each of them representing an arbitrarily small square in plan, and assume that the elementary segments are hinged along the edges. Then formula (1) will yield the frequencies of these elementary shells. Variations in the values of R_1 and R_2 will now cease to have any effect and, assuming that the boundary conditions exert no appreciable influence on the frequency distribution density, we shall be able to follow Courant's example, compute the frequency distribution density for an elementary shell and integrate it over the whole neutral surface. This will cause the ordinates of the graphs in question to become functions of x_1 and x_2, and the integration will have to be performed over the region occupied by the shell. It can easily be shown that the infinite discontinuities now become integrable and vanish, whenever R_1 and R_2 deviate from their constant values.

Note. The above scheme due to Courant is based essentially on the assertion that the frequency distribution density is independent (in the first approximation) of the boundary conditions. This assertion has been shown to hold for a wide range of equations, excluding however the equations of the theory of oscillation of shells.

Degeneracy of the dynamic edge effect means that the boundary conditions become more important. Additional discussion is therefore needed for the Courant's method before it can be used within the region of degeneracy.

We may assume that any appreciable concentration of frequencies can remain sharply defined only in the case of constant curvature shells. The larger the variations in R_1 and R_2, the stronger the smearing out effect of the points of concentration. This aspect might be interesting to investigate, as in the general case perhaps we ought to speak of the regions of concentration and consider the point (or points) to be due to the degeneracy caused by the decrease in the domain of variation of R_1 and R_2. The astmptotic method [5–11] must be used here with great care, since, as we have shown before, it has no provision for taking the variations in R_1 and R_2 into account.

A shallow shell shows little variation in curvature. Points of concentration may appear on it, provided that it is very thin.

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